identity

Lemma Every decomposition of ε in 2-cycles has an even number of 2-cycles.

proof of lemma:

2-cycles

Goal: use induction to show that if $\varepsilon = \beta_1 \cdots \beta_n$, then n even.

Base case: ($n = 1$ or 2)

$\varepsilon \neq (a\,b)$ because $(a\,b)$ doesn't fix a or b.

if $\varepsilon = \beta_1 \beta_2$, 2 is even, so okay.

e.g. $(1\,2)(1\,2)$

induction hypothesis

Now, suppose that if ε is a product of m 2-cycles where $m < n$, then m is even.

Finally, suppose that $\varepsilon = \beta_1 \cdots \beta_n$. NTS n even

induction step.

2-cycles

Express

$$\varepsilon = (\quad)(\quad)\cdots\underbrace{(\quad)}_{\beta_{n-1}}\underbrace{(a\ b)}_{\beta_n}$$

There are four possibilities:

Goal: shuffle a to the left...

	$\beta_{n-1}\beta_n$	replace with	
1.	$(a\ b)(a\ b)$	ε	
2.	$(a\ c)(a\ b)$	$(a\ b)(b\ c)$	$\rightsquigarrow (a\ b\ c)$
3.	$(b\ d)(a\ b)$	$(a\ d)(d\ b)$	$\rightsquigarrow (a\ d\ b)$
4.	$(c\ d)(a\ b)$	$(a\ b)(c\ d)$	

Result of replacement: a appears in β_{n-1} but not β_n.

In case (1), get $\varepsilon = \beta_1 \cdots \beta_n = \beta_1\beta_2 \cdots \beta_{n-2}$

↳ Done because $n-2$ is even by induction hypothesis, so n is even as well ✓

For the remaining three cases, repeat process with

Claim Must eventually find $\beta_i \beta_{i+1} = (ax)(ax)$

for some i and some x.

Why? If not, move a all the way to β_1:

$$\varepsilon = (ax)\underbrace{(\ )\cdots(\ )}_{\text{no } a \text{ here.}}$$

where a appears only in the first transposition.

But than $\varepsilon(a) = x$

a contradiction because $\varepsilon(a) = a$.

Thus $\beta_i \beta_{i+1} = (ax)(ax)$ for some i and some x. So

replace $\beta_i \beta_{i+1}$ with ε to see that

ε is a product of $\overset{<n}{n-2}$ 2-cycles. By induction

$n-2$ is even, so n is even ✓