Group Isomorphisms

$\leftarrow$ "same shape"

Consider:

$\boxed{D_3}$

2nd ↓ \ 1st →	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_0	R_0	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_0	R_{240}	F_3	F_1	F_2
R_{240}	R_{240}	R_0	R_{120}	F_2	F_3	F_1
F_1	F_1	F_2	F_3	R_0	R_{120}	R_{240}
F_2	F_2	F_3	F_1	R_{240}	R_0	R_{120}
F_3	F_3	F_1	F_2	R_{120}	R_{240}	R_0

$R_0 \to \varepsilon$

$R_{120} \to (123)$

$R_{240} \to (132)$

$F_1 \to (12)$

$F_2 \to (23)$

$F_3 \to (13)$

$\boxed{S_3}$

2nd ↓ \ 1st →	ε	(123)	(132)	(12)	(23)	(13)
ε	ε	(123)	(132)	(12)	(23)	(13)
(123)	(123)	(132)	ε	(13)	(12)	(23)
(132)	(132)	ε	(123)	(23)	(13)	(12)
(12)	(12)	(23)	(13)	ε	(123)	(132)
(23)	(23)	(13)	(12)	(132)	ε	(123)
(13)	(13)	(12)	(23)	(123)	(132)	ε

S_3 and D_3 are isomorphic: same group structure, just labelled differently.

a.k.a. function

Defn Sps. G_1 and G_2 are groups. A map $\varphi: G_1 \to G_2$ is a <u>group isomorphism</u> if

- φ 1-1 and onto (bijection ... a relabelling)
- $\varphi(ab) = \varphi(a)\varphi(b)$.

group operation in G_1

group operation in G_2.

"φ preserves the group operation"

If such a map exists between G_1 and G_2, we say

G_1 and G_2 are <u>isomorphic</u>, denoted: $G_1 \approx G_2$.

Idea: isomorphisms give a concrete way to say when two groups are the same in essence.

This concept shows up throughout mathematics:

- invertible linear transformations in linear algebra
- isometry in geometry
- homeomorphism in topology/analysis

Note: if G_1 and G_2 are finite and same size, by an exercise it suffices to check 1-1 or onto.