Again: Isomorphisms $\varphi: G_1 \to G_2$ are simply relabellings...

underlying group structures of G_1 and G_2 are exactly the same.

So: if $\varphi: G_1 \to G_2$ is an isomorphism, we have the following:

↳ φ 1–1, onto

$\varphi(ab) = \varphi(a)\varphi(b)$.

Properties of Isomorphisms

sps $a \in G_1$

$\varphi(a)e_2 = \varphi(a) = \varphi(ae_1) = \varphi(a)\varphi(e_1) \Rightarrow e_2 = \varphi(e_1)$.

1. $\varphi(e_1) = e_2$ (identity maps to identity)

2. $\varphi(a^n) = [\varphi(a)]^n$ for all $a \in G_1$ and every power of n.

For the positive powers, prove by induction:

Base case: $n=2$: $\varphi(a^2) = \varphi(aa) \overset{\text{defn isom.}}{=} \varphi(a)\varphi(a) = [\varphi(a)]^2$. ✓

Induction hypothesis: sps. $\varphi(a^k) = [\varphi(a)]^k$.

Induction step: $\varphi(a^{k+1}) = \varphi(a^k a) \overset{\varphi \text{ isom.}}{=} \varphi(a^k)\varphi(a) \overset{\text{ind. hyp}}{=} [\varphi(a)]^k \varphi(a)$

$= [\varphi(a)]^{k+1}$.

Negative powers follow from:

3. $\varphi(a^{-1}) = [\varphi(a)]^{-1}$

φ is om. property 1

Why? $\varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(e_1) = e_2$. ✓

4. $ab = ba \Leftrightarrow \varphi(a)\varphi(b) = \varphi(b)\varphi(a)$ so:

G_1 is abelian $\Leftrightarrow$ G_2 is abelian.

5. $G_1 = \langle a \rangle \Leftrightarrow G_2 = \langle \varphi(a) \rangle$ so:

G_1 is cyclic $\Leftrightarrow G_2$ is cyclic and

a generates $G_1 \Leftrightarrow \varphi(a)$ generates G_2

6. $|a| = |\varphi(a)|$

Also: G_1 has r elements of order $n \Leftrightarrow G_2$ has r elements of order n.

* handy for showing groups are not isomorphic.

subset

7. Sps $K \subseteq G_1$.

set

Define $\varphi(K) = \{ \varphi(a) \mid a \in K \} \subseteq G_2$.

Then

K a subgroup of $G_1 \Leftrightarrow \varphi(K)$ a subgroup of G_2.