Ex let $a \in G$. (a fixed)

domain codomain

Consider $\varphi_a : G \to G$.

x variable

$$\varphi_a(x) = axa^{-1}$$ (conjugation by a)

a constant

For any choice of $a \in G$, φ_a is an isomorphism.

1-1? Sps $\varphi_a(x) = \varphi_a(y)$ (NTS: $x = y$)

Then $axa^{-1} = aya^{-1}$

So $xa^{-1} = ya^{-1}$ (cancel a on left)

and $x = y$. ✓ (cancel a^{-1} on right)

codomain domain

onto? Sps $z \in G$. (Need: $x \in G$ s.t. $\varphi_a(x) = z$.)

Consider $x = a^{-1}za$. ← $\in G$ b/c $a, z \in G$.

Then $\varphi(x) = \varphi(a^{-1}za) = aa^{-1}zaa^{-1} = z$. ✓

operation-preserving? Sps. $x, y \in G$.

$$\varphi_a(xy) = axya^{-1} = ax\underbrace{a^{-1}a}_{e}ya^{-1} = \varphi_a(x)\varphi_a(y). \checkmark$$

defn of φ_a

<u>Defn</u> An isomorphism from G to itself is called an <u>automorphism</u>.

e.g. $\varphi_a : G \to G$ an automorphism

an algebraic symmetry of G ... analogous to an element of D_3 acting on Δ.

binary operation : addition

<u>Ex</u> $\varphi : \mathbb{Z} \to \mathbb{Z}$

$$\varphi(x) = -x$$

1-1? ✓ onto? ✓

operation preserving? $\varphi(\underbrace{x+y}_{\text{domain } \mathbb{Z}}) \overset{\text{apply } \varphi}{=} -(x+y)$

$= -x-y$

$= (-x)+(-y)$

$= \underbrace{\varphi(x)+\varphi(y)}_{\text{codomain } \mathbb{Z}}$. ✓

conjugation

Defn Given an element $a \in G$, the <u>inner automorphism induced by a</u> is the map

$$\varphi_a : G \to G$$

$$\varphi_a(x) = axa^{-1}.$$

we saw above that this is an isomorphism

ie $\varphi_a(x) = \varphi_b(x)$ $\forall x \in G$.

Note: can have $a \neq b$ but $\varphi_a = \varphi_b$ as functions.

Extreme example: Sps. G is abelian. For any $a \in G$

$$\varphi_a(x) = axa^{-1} \overset{\text{abelian}}{=} aa^{-1}x = x \quad \text{for all } x \in G.$$

so φ_a is the identity map for every $a \in G$.

Notation: $\mathrm{Aut}(G) = \{\text{automorphisms of } G\}$.

conjugation

$\mathrm{Inn}(G) = \{\text{inner automorphisms of } G\}$

Thm: $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are groups under composition. (exercise)