recall intro to permutation groups.

Thm (Cayley's Thm) Every group G is isomorphic to a group of permutation.

Note: If $|G| = n$, we'll see G is isomorphic to a subgroup of S_n.

if G finite

↳ Idea: G is isomorphic to a group of permutations of its own elements

$G = A$ (see intro to permutation gps)

proof of Cayley's thm:

Strategy: 1. Construct a group $\bar{G}$ of permutations based on G.

2. Show $G \approx \bar{G}$.

isomorphic

1-1, onto function $A \to A$.

1. For each $g \in G$, construct a permutation L_g of the $\underbrace{\text{set of elements of } G}_{A}$ by

$$L_g : G \to G$$

x variable

$$L_g(x) = gx$$

g fixed

(left-translation by g ... consider Cayley table)

Recall: L_g is a permutation:

(NTS: $x=y$)

1-1? Sps $L_g(x) = L_g(y)$.

Then by defn L_g, $gx = gy$. So $x = y$. (cancel g on left)

onto?

Sps $z \in G$. (Need: $x \in G$ s.t. $L_g(x) = z$.)

Consider $x = \underbrace{g^{-1}z}_{\in G.}$ ✓

Then $L_g(x) = L_g(g^{-1}z) = gg^{-1}z = z$. ✓

So yes! L_g is a permutation.

subset

Let $\bar{G} = \{ L_g \mid g \in G \} \subseteq$ permutations of elts of G.

Then $\bar{G}$ is a subgroup of the group of permutations of the elements of G, as follows.

composition

• closure: sps $L_g, L_h \in \bar{G}$. Then $L_g L_h \in \bar{G}$ because

$L_g L_h = L_{gh}$ because for all $x \in G$,

↳ gh is an elt of G, so $L_{gh} \in \bar{G}$

want these equal as functions.

So as functions $L_g L_h = L_{gh}$. ✓ *

$$* \quad (L_g L_h)(x) = L_g(L_h(x)) = L_g(hx) = g(hx) = (gh)x = L_{gh}(x)$$

defn composition — defn of L_h — defn of L_g — assoc in G — defn gh.

• identity?

$\varepsilon(x) = x$

Consider L_e. Then $L_e(x) = ex = x$

identity in G — defn of L_e

So L_e acts as identity map. ✓

• $(L_g)^{-1} = L_{g^{-1}}$

↳ $g^{-1} \in G$, so $L_{g^{-1}} \in \bar{G}$.

$$(L_g L_{g^{-1}})(x) = L_g(L_{g^{-1}}(x)) = L_g(g^{-1}x) = gg^{-1}x = x. \checkmark$$

So $L_g L_{g^{-1}}$ acts as identity.

Thus $(L_g)^{-1} = L_{g^{-1}}$ ✓

So $\bar{G} = \{ L_g \mid g \in G \}$ is a group of permutations.