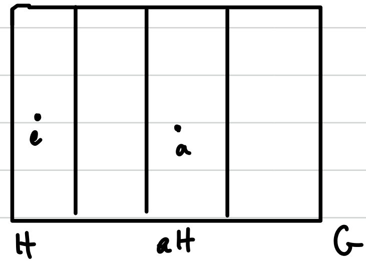
Properties of cosets:



Sps. $H \leq G$. Let $a, b \in G$

1. $a \in aH$ (subset of G.)

(since $e \in H$, $ae \in aH$, i.e. $a \in aH$.)

2. Either $aH = bH$ or $aH \cap bH = \phi$. (empty set)

↳ ie. cosets of H partition G.

proof: Sps $aH \cap bH \neq \phi$. (NTS: $aH = bH$ – $aH \subset bH$ and $bH \subset aH$)

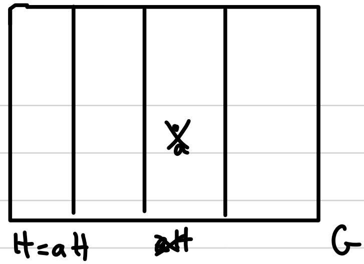
Since $aH \cap bH \neq \phi$, sps $g \in aH \cap bH$. (always include this type of quantifier)

This implies $g = ah_1 = bh_2$ for some $h_1, h_2 \in H$

Let $ah_3 \in aH$. (generic elt of aH)

Then $ah_3 = a\underbrace{h_1 h_1^{-1}}_{e} h_3 = bh_2 h_1^{-1} h_3 = bh_4$ where $h_4 = h_2 h_1^{-1} h_3 \in H$. $\in bH$. So $aH \subset bH$.

Similarly, $bH \subset aH$, so $aH = bH$.



3. $aH = H \iff a \in H$.

($\Rightarrow$) Sps $aH = H$.

Then since $a \in aH$, and $aH = H$, conclude $a \in H$.
property 1

($\Leftarrow$) Sps. $a \in H$. Since $a \in aH$, $aH \cap H \neq \emptyset$, thus $aH = H$.
property 1   property 2

how to tell when cosets are equal. "litmus test"

* 4. $aH = bH \iff a^{-1}b \in H$ (or $b^{-1}a \in H$)

($\Rightarrow$) Sps $aH = bH$.

defn of aH

$b \in bH \Rightarrow b \in aH \Rightarrow b = ah$ for some $h \in H$. $\Rightarrow a^{-1}b = h$ for some $h \in H$, i.e. $a^{-1}b \in H$

always include

($\Leftarrow$) Sps $a^{-1}b \in H$. So $a^{-1}b = h$ for some $h \in H$.

So $b = ah$ for some $h \in H$

i.e. $b \in aH$ (by defn of aH)

But $b \in bH$ (property 1)

So $aH \cap bH \neq \emptyset$. Thus $aH = bH$, by property 2.