**Thm** If $(g,h) \in G \oplus H$,

$$|(g,h)| = \text{lcm}(|g|, |h|)$$

↙ note similarity with proof about order of product of disjoint cycles in $S_n$.

proof: Let $t = |(g,h)|$ and $s = \text{lcm}(|g|, |h|)$

(NTS: $t = s$. Show: $s|t$ and $t|s$.)

To see that $s|t$, notice that since $t = |(g,h)|$,

$$(e_G, e_H) = (g,h)^t = (g^t, h^t)$$

So $g^t = e_G$ and $h^t = e_H$. Thus $|g| \mid t$ and $|h| \mid t$.

So $t$ is a common multiple of $|g|$ and $|h|$.

By an earlier exercise, $s|t$.

$\text{lcm}(|g|, |h|)$.

$\swarrow \text{lcm}(|g|,|h|)$

OTOH, $s$ is a common multiple of $|g|$ and $|h|$, so

$$(g,h)^s = (g^s, h^s) = (e_G, e_H).$$

$\swarrow |(g,h)|$

Therefore $|(g,h)| \mid s$, i.e. $t \mid s$.

We conclude that $t = s$, i.e. $|(g,h)| = \text{lcm}(|g|,|h|)$.

Note: can use induction to show that if $(g_1, \ldots, g_n) \in G_1 \oplus \cdots \oplus G_n$

then

$$|(g_1, \ldots, g_n)| = \text{lcm}(|g_1|, \ldots, |g_n|).$$