not cyclic $\qquad\qquad\qquad$ cyclic

Recall: $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ vs. $\mathbb{Z}_2 \oplus \mathbb{Z}_3$

Thm Sps G, H are cyclic with $|G| = m$ and $|H| = n$. Then

$$G \oplus H \text{ cyclic} \iff \gcd(m, n) = 1.$$

order mn.

proof: $(\Leftarrow)$ Sps $\gcd(m,n) = 1$.

generators of G and H.

Sps $G = \langle g \rangle$ and $H = \langle h \rangle$

Then $|(g, h)| = \text{lcm}(|g|, |h|) = \text{lcm}(m, n)$.

$\gcd(m,n) = 1$

Since m and n have no common factors,

$\text{lcm}(m, n) = mn$.

Since $|G \oplus H| = mn$, $G \oplus H$ is cyclic, generated by (g, h).

generators of G and H.

($\Rightarrow$) OTOH, sps. $G \oplus H$ is cyclic, generated by (g', h').

So $|(g', h')| = mn$ and mn is the smallest power s.t. $(g', h')^{mn} = (e_G, e_H)$.

Let $d = \gcd(m, n)$. (NTS: $d = 1$)

Then $\frac{m}{d} \in \mathbb{Z}$ and $\frac{n}{d} \in \mathbb{Z}$. Thus

$$(g', h')^{\frac{mn}{d}} = \left(((\underbrace{(g')^{\overset{|G|}{m}}}_{e_G})^{\frac{n}{d}}, ((\underbrace{(h')^{\overset{|H|}{n}}}_{e_H})^{\frac{m}{d}} \right) = (e_G, e_H)$$

(because $|g'| \,\big|\, |G|$ and $|h'| \,\big|\, |H|$)

Thus, $d = 1$ because mn is the smallest power we can raise (g', h') to to get the identity. ✓

Note: this extends to the product of many groups:

Sps. $G_1, G_2, \ldots, G_n$ are finite cyclic.

Then $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ cyclic

$$\Longleftrightarrow$$

$$\gcd(|G_i|, |G_j|) = 1 \text{ when } i \neq j.$$

pairwise relatively prime

Ex. $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$ ⟵ cyclic

Ex. $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$ ⟵ not cyclic