Recall:

<u>Thm</u> Sps $H \trianglelefteq G$, $K \trianglelefteq G$, $G = HK$, and $H \cap K = \{e\}$. Then

$$G \approx H \oplus K = \{(h,k) \mid h \in H, k \in K\}.$$

proof:

1. Show every element $g \in G$ can be expressed uniquely as $g = hk$ where $h \in H$ and $k \in K$.

2. Define $\varphi : G \to H \oplus K$ by $\varphi(g) = \varphi(hk) = (h,k)$ and show φ is an isomorphism.

Lemma Under given conditions, for any

$h \in H$, $k \in K$, $hk = kh$, i.e. $hkh^{-1}k^{-1} = e$.

elts of H and K commute

$H \trianglelefteq G$
$K \trianglelefteq G$.
$G = HK$
$H \cap K = \{e\}$

proof of lemma:

$H \trianglelefteq G$, $K \trianglelefteq G$ $\Rightarrow$ $hkh^{-1} \in K$ $\quad kh^{-1}k^{-1} \in H$

normal subgroup test.

so $(hkh^{-1})k^{-1} \in K$

same

and $h(kh^{-1}k^{-1}) \in H$

But $H \cap K = \{e\}$ so $hkh^{-1}k^{-1} = e$. ✓

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Now, for step 1 of proof of theorem, let $g \in G$.

Since $G = HK$, $g = hk$ some $h \in H$, $k \in K$.

uniqueness proof

suppose $g = h_1k_1 = h_2k_2$. (NTS: $h_1 = h_2$, $k_1 = k_2$).

Then $h_1k_1k_2^{-1}h_2^{-1} = e$.
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By the lemma, h_1 commutes will all elements of K,

so this becomes

$$(k_1 k_2^{-1})(h_1 h_2^{-1}) = e$$

$$\text{Thus} \quad \overset{\in K}{k_1 k_2^{-1}} = \overset{\in H}{h_2 h_1^{-1}} = e \quad \text{since } H \cap K = \{e\}.$$

Therefore $h_1 = h_2$ and $k_1 = k_2$. ✓

Now, for step 2,

Define map

$$\varphi : G \to H \oplus K \text{ by}$$

$$\varphi(g) = \varphi(hk) = (h, k).$$

Since every element of G can be written uniquely as a product $g = hk$, φ is a well-defined function on G.

NTS: φ an isomorphism: 1-1, onto, operation-preserving.

1-1: Sps $\varphi(g) = \varphi(g') = (h,k)$.

Then $g = hk = g'$. ✓

onto: Sps $(h,k) \in H \oplus K$.

Then $(h,k) = \varphi(hk)$ and $hk \in G$. ✓

operation-preserving:

$$
\begin{aligned}
\varphi(g_1 g_2) &= \varphi((h_1 k_1)(h_2 k_2)) \\
&= \varphi(h_1 k_1 h_2 k_2) \\
&= \varphi(h_1 h_2 k_1 k_2) && \text{(elements of } H \text{ and } K \text{ commute) lemma} \\
&= (h_1 h_2, k_1 k_2) && \text{(apply } \varphi) \\
&= (h_1, k_1)(h_2, k_2) && \text{(operation in } H \oplus K) \\
&= \varphi(g_1)\varphi(g_2) && (g_1 = h_1 k_1,\ g_2 = h_2 k_2 \ldots \text{ defn of } \varphi).
\end{aligned}
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