Normal Subgroups

<u>Defn</u> Sps $H \leq G$. H is a <u>normal</u> subgroup, denoted if $aH = Ha$ for all $a \in G$.

Note: this says $aH = Ha$ as sets. So: for each $h_1 \in H$ there exists $h_2 \in H$ s.t. $ah_1 = h_2 a$. (and vice versa). Doesn't say $H \subseteq Z(G)$, ← center

ie not saying $ah = ha$ $\forall h \in H$, $a \in G$.

too strong.

<u>Thm</u> (Normal subgroup test)

$H \leq G$ is normal if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.

i.e. if and only if $xhx^{-1} \in H$ for all $h \in H$ and for all $x \in G$.

easy to check

(proof to come)

Ex. $\overset{R}{=}\{\text{rotations}\} \triangleleft D_3$

flips and rotations

Why?

Sps $r \in R$. sps $x \in D_3$.

If x a rotation, then xrx^{-1} is a rotation, so in R. ✓

If x a flip, the $xrx^{-1} = xrx$ = a rotation b/c even # of flips.

$x^{-1} = x$

ie. $xrx^{-1} \in R$. ✓

det A = 1 det A ≠ 0

Ex. $SL(2,\mathbb{R}) \triangleleft GL(2,\mathbb{R})$

Sps $S \in SL(2,\mathbb{R})$ and $X \in GL(2,\mathbb{R})$.

Check: $XSX^{-1} \in SL(2,\mathbb{R})$?

$\det(XSX^{-1}) = \det X \det S \det X^{-1}$

$= \det X \det S \dfrac{1}{\det X}$

$= \det S = 1$ b/c $S \in SL(2,\mathbb{R})$ ✓

$\left\{ \begin{array}{l} XX^{-1} = I \\ \Downarrow \qquad \uparrow \det 1 \\ \det X^{-1} = \dfrac{1}{\det X} \end{array} \right.$

So $XSX^{-1} \in SL(2,\mathbb{R})$ ... So $SL(2,\mathbb{R}) \triangleleft GL(2,\mathbb{R})$ ✓

Ex. Sps G abelian. For any $H \leq G$, $H \trianglelefteq G$.

subgroup

Why? Sps $h \in H$, $x \in G$.

Then $xhx^{-1} = xx^{-1}h = eh = h \in H$.

So by normal subgroup test, $H \trianglelefteq G$.