Defn: H normal if $aH = Ha$ for all $a \in G$.

Thm (Normal subgroup test)

$H \leq G$ is normal if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.

i.e. if and only if $xhx^{-1} \in H$ for all $h \in H$ and for all $x \in G$.

proof: We know that for any $x \in G$,

$$xH = Hx \iff xHx^{-1} = H. \quad \text{(property of cosets)}$$

So $H \trianglelefteq G \Rightarrow xHx^{-1} \subseteq H$ for all $x \in G$.

↳ weaker statement than we could make here

OTOH, sps. $xHx^{-1} \subseteq H$ for all $x \in G$.

Fix $a \in G$. (NTS: $aH = Ha$.)

Then $aHa^{-1} \subseteq H$ means that for all $h_1 \in H$ there exists $h_2 \in H$ such that $ah_1a^{-1} = h_2$

Thus for all $h_1 \in H$, there exists $h_2 \in H$ such that $ah_1 = h_2a$ Thus $aH \subseteq Ha$.

Now, let $x = a^{-1}$. Then by hypothesis, $a^{-1}Ha \subseteq H$.

Thus for all $h_1 \in H$ there exists $h_2 \in H$ such that

$$a^{-1}h_1a = h_2.$$

ie. for all $h_1 \in H$ there exists $h_2 \in H$ such that

$$h_1a = ah_2.$$

This says $Ha \subseteq aH$.

Therefore, for any $a \in G$, $aH = Ha$, so $H \trianglelefteq G$.