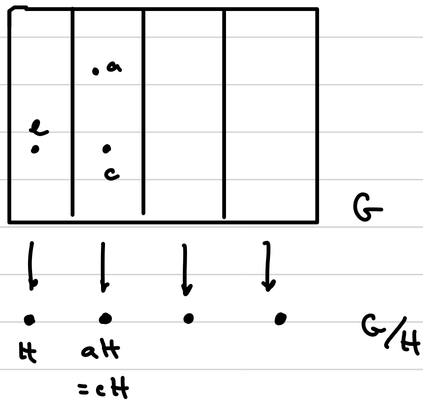
Recall:

Thm Sps. $H \triangleleft G$. The set of distinct left cosets, denoted G/H, is a group under the operation



$$(aH)(bH) = abH.$$

proof:

First, we must show that the binary operation

$$G/H \times G/H \to G/H$$

given by $aH\, bH \longmapsto abH$

is well-defined as a function.

relies on fact that H is normal.

Then we can check closure, associativity, identity, inverses.

Sps $aH = cH$ and $bH = dH$. (NTS: $abH = cdH$)

properties of cosets

Then $c^{-1}a \in H$ and $d^{-1}b \in H$.

So, show: $(cd)^{-1}ab \in H$.

Consider:

$$(cd)^{-1}ab = d^{-1}\underbrace{c^{-1}a}_{\in H}b = d^{-1}hb \quad \text{for some } h \in H.$$

so $bH = Hb$

But H is normal so for all $h \in H$, there exists h' such that $hb = bh'$.

some $h'' \in H$

Thus

$$d^{-1}hb = \underbrace{d^{-1}b}_{\in H}h' = h''h' \in H.$$ So $(cd)^{-1}ab \in H$, thus $cdH = abH$.

Therefore,

so the binary operation is well-defined.

Finally,

closure: given $aH, bH \in G/H$,

$a \in G$ and $b \in G$ so $ab \in G$

$abH \in G/H$

Thus ~~$ab \in G$~~ so G/H closed ✓
under binary operation.

associativity:

$(aH)(bHcH) = (aH)(bcH) = a(bc)H$

$= (ab)cH = (abH)(cH) = (aHbH)(cH)$. ✓

associativity in G

identity: Consider eH (a.k.a. H)

Then for any $aH \in G/H$,

$(aH)(eH) = aeH = aH = eaH = (eH)(aH)$.

So eH is identity.

inverses: Sps $aH \in G/H$. Consider $a^{-1}H$ ← so, inverse of aH. ✓

Then $(aH)(a^{-1}H) = aa^{-1}H = eH$ ← identity.