# Internal Direct Products

We have considered external direct products $G_1 \oplus G_2$.

Now, given $G$, can we find subgroups $H, K \leq G$ such that $G \approx H \oplus K$? i.e. is $G$ an internal direct product of $H$ and $K$?

Goal: Find conditions under which this can happen.

Given, $H, K \leq G$,

Define the following set:

$$HK = \{ hk \mid h \in H, k \in K \}$$

subset

in general, $HK \subseteq G$ but it is not necessarily all of $G$ or a subgroup of $G$.

extreme example: $H = K = \{e\}$.

$\rightarrow$ so $G \subseteq HK$

We say $G = HK$ if they are equal as sets, i.e. every element $g \in G$ can be expressed

$$g = hk$$

for some $h \in H$, $k \in K$.

Note: If $G = HK$, expression for $g$ might not be unique: could have

$$g = h_1 k_1 = h_2 k_2 \quad \text{where } h_1 \neq h_2 \text{ and } k_1 \neq k_2.$$

$\hookrightarrow$ extreme example: $H = K = G$.

In order for $G \cong H \oplus K$, we will need conditions to ensure that for every $g \in G$ there exists a unique pair $h, k$ such that

$$g = hk.$$